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LETTER TO THE EDITOR

On spin systems related to the Temperley–Lieb algebra

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Abstract
The spectrum of the transfer matrices constructed from the spectral parameter dependent Temperley–Lieb R-matrix is found using functional relations identical to those of the spin $1/2$ XXZ-magnet.

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It is well known that the Temperley–Lieb algebra [1, 2] gives rise to a constant solution of the Yang–Baxter equation (see e.g. [3] and references therein)

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23} \tag{1}$$

where $\check{R}_{12} = \check{R} \otimes I$, $\check{R}_{23} = I \otimes \check{R}$, and $\check{R}$ is a constant R-matrix ($\check{R} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$) defined by the Temperley–Lieb idempotent X

$$\check{R} = qI + X \qquad X^2 = -\left(q + \frac{1}{q}\right) X \tag{2}$$

satisfying the Hecke condition (we suppose that q is not a root of unity)

$$\check{R}^2 = \left(q - \frac{1}{q}\right)\check{R} + I. \tag{3}$$

The Temperley–Lieb algebra (TL_N) has $N-1$ generators $\{1, X_1, X_2, \ldots, X_{N-1}\}$ subject to the relations ($d = -\nu(q) := -(q + 1/q)$)

$$\begin{aligned} &X_k^2 = dX_k \\ &X_k X_{k\pm 1} X_k = X_k \\ &X_j X_k = X_k X_j \qquad |j-k| > 1. \end{aligned} \tag{4}$$

Using the FRT formalism [4, 5], a matrix realization of TL_N and the R-matrix (2) one can define a quantum group $\mathcal{A}(R)$ while the Baxterization procedure results in a spectral parameter dependent R-matrix

$$\check{R}(u) = u\check{R} - \frac{1}{u}\check{R}^{-1}. \tag{5}$$

Due to the Hecke condition (3) this R-matrix has the regularity property [6]

$$\check{R}(1) = \left(q - \frac{1}{q}\right) I \tag{6}$$

although usually ($n > 2$) it is not a quasiclassical solution of the YBE. Hence, an integrable quantum spin system constructed from the *L*-operator

$$L(u) = R(u) = \mathcal{P}\check{R}(u) = \left(uq - \frac{1}{uq}\right)\mathcal{P} + \left(u - \frac{1}{u}\right)\mathcal{P}X \tag{7}$$

where $\mathcal{P} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ is the permutation operator, has a local Hamiltonian with nearest-neighbour interaction

$$H = \sum_{k=1}^{N} X_k \tag{8}$$

subject to the periodic boundary condition: $X_N := X_{N1} \in \mathrm{End}\big(\mathbb{C}^n_N \otimes \mathbb{C}^n_1\big)$ or free ends.

Particular realizations of these spin systems can be found in a variety of papers (see [7–11] and references therein) as well as the spectra of some spin Hamiltonians ($n = 3$) obtained by coordinate Bethe ansätze (see, e.g., [12, 13]).

In this letter, we analyse the spectrum of the Hamiltonian (8) and corresponding transfer matrix $t^{(1)}(u)$ using the fusion procedure and functional relations [6] for transfer matrices $t^{(m)}(u)$ in higher representations V_m of quantum algebra dual to $\mathcal{A}(R)$. The found spectrum coincides with the spectrum of the XXZ-model, however its degeneracy and the Bethe states depend heavily on a realization of the idempotent $X \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$. Solutions of the reflection equation describing non-periodic boundary conditions are also given.

We fix a local realization of the TL_N algebra $\{1, X_1, X_2, \ldots, X_{N-1}\}$ in the (quantum) space

$$\mathcal{H}_N = \otimes_{k=1}^{N} \mathbb{C}^n_k. \tag{9}$$

This realization is defined by an invertible $n \times n$ matrix b, $\bar{b} := b^{-1}$. This matrix can be treated as a vector of the n^2-dimensional space $\mathbb{C}^n \otimes \mathbb{C}^n$ with components b_{ij}, $i, j = 1, 2, \ldots, n$, and the rank 1 matrix $X \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ (the idempotent)

$$X = b \otimes \bar{b} \qquad X_{ab;cd} = b_{ab}\bar{b}_{cd} \tag{10}$$

is a generator of TL_N with X_k acting nontrivially on two factors $\mathbb{C}^n_k \otimes \mathbb{C}^n_{k+1}$ of $\mathcal{H}_N$.

The Hecke or TL_N algebra parameter q entering $\check{R}$ is defined by the matrix b

$$\nu(q) := \left(q + \frac{1}{q}\right) = -\sum_{a,b} b_{ab}\bar{b}_{ab} = -\mathrm{Tr}\, b^t \bar{b}.$$

The *R*-matrix $R(u) = \mathcal{P}\check{R}(u)$ and the *L*-operator (7) satisfy the YBE with spectral parameter

$$R_{a_1a_2}(u/w)L_{a_1j}(u)L_{a_2j}(w) = L_{a_2j}(w)L_{a_1j}(u)R_{a_1a_2}(u/w) \tag{11}$$

where the subscripts a_1, a_2 refer to the two auxiliary spaces while j refers to the quantum space $\mathbb{C}^n_j$ at site j [5, 6]. It is easy to see that the *R*-matrix (5) (this is a braid group form)

$$\check{R}_{a_1a_2}(u) = \left(uq - \frac{1}{uq}\right) I_{a_1a_2} + \left(u - \frac{1}{u}\right) X_{a_1a_2} = \omega(uq)(I - P) - \omega(u/q)P \tag{12}$$

has two degeneracy points $u = q^{\pm 1}$

$$\check{R}(q^{-1}) = \omega(q^2)P \qquad \check{R}(q) = \omega(q^2)(I - P) \tag{13}$$

where $P = -X/\nu(q)$ is the rank 1 projector $P^2 = P$, and $\nu(q) = q + 1/q$, $\omega(q) := q - 1/q$.

Using the standard formalism of the quantum inverse scattering method (see, e.g., [5, 6]) we define the monodromy matrix

$$T(u) = L_{aN}(u)L_{aN-1}(u)\cdots L_{a1}(u) = \prod_{j=1}^{N} L_{aj}(u) \tag{14}$$

and the transfer matrix (the generating function of mutually commuting integrals)

$$t(u) = \mathrm{Tr}\, T(u) = \mathrm{Tr}_{(a)} \prod_{j=1}^{N} L_{aj}(u) \tag{15}$$

as operators on the space $\mathbb{C}_a^n \otimes \mathcal{H}_N$ and $\mathcal{H}_N$ correspondingly. The monodromy matrix $T(u)$ satisfies relation (11). Various properties of $T(u)$ and $t(u)$ follow from the structure of the Temperley–Lieb R-matrix (L-operator) (7). For example, due to the regularity (6) the transfer matrix $t(u)$ at $u = 1$ is the right shift operator

$$t(1) = \omega(q)^N \mathcal{P}_{12}\mathcal{P}_{23}\cdots\mathcal{P}_{N-1N} = \omega(q)^N U \tag{16}$$

and due to the degeneracy at $u = q^{-1}$

$$t(q^{-1}) = (-\omega(q))^N \mathcal{P}_{N-1N}\cdots\mathcal{P}_{23}\mathcal{P}_{12} = (-\omega(q))^N U^{-1} \tag{17}$$

is the left shift of the quantum space $\mathcal{H}_N$.

It is not difficult to find a bare vacuum or a reference state Ω used in the algebraic Bethe ansatz of the quantum inverse scattering method. The transfer matrix $t(u)$ (26) is the sum of 2^N terms two of which are the shift operators (16), (17)

$$t(u) = \mathrm{Tr}\, T(u) = \omega(uq)^N U + \sum_j Y_j + \omega(u)^N U^{-1}. \tag{18}$$

Other terms Y_j have at least one operator factor $\mathcal{P}_{ak}X_{ak}\mathcal{P}_{ak-1}$ which yields

$$(\mathcal{P}_{ak}X_{ak}\mathcal{P}_{ak-1})_{i_a i_k i_{k-1} j_a j_k j_{k-1}} = b_{i_k i_a}\bar{b}_{j_{k-1} j_k}\delta_{i_{k-1} j_a} \tag{19}$$

as matrix entries of this operator at the space $\mathbb{C}_a^2 \otimes \mathbb{C}_k^2 \otimes \mathbb{C}_{k-1}^2$ under the trace over the auxiliary space $\mathbb{C}_a^2$. Taking a homogeneous vector $\Omega := \otimes_1^N w$ invariant under the shift

$$U^{\pm 1} \otimes_1^N w = \otimes_1^N w = \Omega \tag{20}$$

combined with the local vectors $w \in \mathbb{C}^n$ satisfying

$$(\bar{b}, w \otimes w) = \sum \bar{b}_{ij} \qquad w_i w_j = 0 \tag{21}$$

one gets an eigenvector of $t(u)$ (15), (18)

$$t(u)\Omega = \Lambda_0(u)\Omega \qquad \Lambda_0(u) = \omega(uq)^N + \omega(u)^N. \tag{22}$$

There are many solutions of equation (21), e.g. if $\bar{b}_{11} = 0$, then one can take $w^t = (1, 0, \ldots, 0)$, and Ω is the state with 'all spins up' $w_i = \delta_{i1}$. However, construction of excited eigenstates by algebraic or coordinate Bethe ansätze depends on the structure of the vector b.

The degeneracy points (13) and the fusion procedure [6, 14] give rise to monodromy and transfer matrices $T^{(m)}(u), t^{(m)}(u) = \mathrm{Tr}_{V_m} T^{(m)}(u), m = 1, 2, \ldots$ in higher dimensional representation spaces V_m of the underlying (dual) quantum group defined by the FRT formalism and higher R-matrices. In particular, denoting the initial (fundamental) representation by $V_1 := \mathbb{C}^n$ the next representation V_2 is $(n^2 - 1)$-dimensional and follows from the Clebsch–Gordan decomposition

$$V_1 \otimes V_1 = V_2 \oplus V_0 \tag{23}$$

with $\dim V_0 = 1$. The corresponding monodromy matrix $T^{(2)}(u)$ is

$$T^{(2)}(u) = (\omega(q^2))^{-1}\check{R}_{12}(q)T_1^{(1)}(uq)T_2^{(1)}(u) = P_+T_1^{(1)}(u)T_2^{(1)}(uq)P_+ \tag{24}$$

where the subscripts denote two auxiliary spaces and the superscripts refer to the representations $V_m, m = 1, 2$. The notation $P_+ = I - P$ is also introduced for the projector on the space V_2 in the CG expansion (23). The projection on the one-dimensional space V_0 yields the value of a multiplicative central element

$$(d(u))^N := (\omega(q^2))^{-1}\check{R}_{12}(q^{-1})T_1^{(1)}(u)T_2^{(1)}(uq) = (\omega(u)\omega(uq^2))^N I. \tag{25}$$

Dimensions of the representation spaces V_m are given by values of the Chebyshev polynomials of the 2D kind ($\dim V_m = p_m(n)$) defined by the recurrence relations:

$$p_{m+1}(x) + p_{m-1}(x) = xp_m(x) \qquad p_0(x) = 1 \quad p_{-1}(x) = 0. \tag{26}$$

However, despite the different dimensions (26) the structure of the Clebsch–Gordan decomposition of tensor products of these representations is identical to the $sl(2)$ case. Hence, one gets for the transfer matrices $t^{(m)}(u)$ in higher dimensional auxiliary spaces V_m functional relations identical to the case of the XXZ-model [6, 15, 16]

$$\begin{aligned} t^{(1)}(u)t^{(1)}(uq) &= (\omega(uq))^N t^{(2)}(u) + (d(u))^N I \\ t^{(1)}(u)t^{(m)}(uq) &= (\omega(uq))^N t^{(m+1)}(u) + (\omega(u))^N t^{(m-1)}(uq^2). \end{aligned} \tag{27}$$

We conclude that the structure of the fusion relations, the analytical properties of the transfer matrix (15) and its eigenvalue (22) on the bare vacuum (20) coincide with those of the spin $1/2$ XXZ-magnet. Hence, according to the analytic Bethe ansatz [17] the spectrum of $t^{(1)}(u)$ of the spin system related to the R-matrix (5) is also the same:

$$\Lambda\big(u; \{v_j\}_1^M\big) = \omega(uq)^N \prod_{j=1}^{M} \frac{\omega(u/qv_j)}{\omega(u/v_j)} + \omega(u)^N \prod_{j=1}^{M} \frac{\omega(uq/v_j)}{\omega(u/v_j)}. \tag{28}$$

An algebraic construction of eigenstates by an algebraic Bethe ansatz (ABA) depends on the vector b_{ij}. In particular, taking for $n = 3, b_{ij} = p^{j-2}\delta_{i4-j}$ one reproduces a deformed spin 1 chain (see, e.g., [12]). In this case a modified ABA can be constructed using the entry $T_{13}(v)$ as an elementary magnon creation operator over the ferromagnetic vacuum state ('all spins up'). The limit $p \to 1$ yields the $sl(2)$-invariant Hamiltonian density $(\mathbf{S}_k, \mathbf{S}_{k+1})^2$ [18]. One can also employ inversion relations to get spectra of the transfer matrices in the thermodynamic limit [9].

A few remarks can be made on solutions to the reflection equation (RE) describing non-periodic boundary conditions preserving integrability [19]. The constant solution of the RE (in the braid group form)

$$\check{R}_{12}K_2\check{R}_{12}K_2 = K_2\check{R}_{12}K_2\check{R}_{12} \tag{29}$$

with $\check{R}$ (2) is given by $n \times n$ matrix K with algebraic entries satisfying characteristic equation [20]

$$qK^2 + c_1K = -\frac{1}{\nu(q)}\big(c_1^2 + qc_2\big)I. \tag{30}$$

The elements c_1, c_2 are central $[c_\alpha, K_{jk}] = 0, \alpha = 1, 2;\ j, k = 1, 2, \ldots, n$. They are given by the quantum trace [4]

$$c_\alpha = \mathrm{Tr}_q\, K^\alpha = \mathrm{Tr}\, b^t\bar{b}K^\alpha = \sum_{i,j,k} b_{ij}\bar{b}_{ik}(K^\alpha)_{kj}. \tag{31}$$

A spectral parameter dependent solution can be obtained by a Baxterization procedure similar to that used for constructing the *R*-matrix (5) (see, e.g., [21]). Then the free end Hamiltonian (8) and an appropriate Sklyanin transfer matrix will be a quantum group invariant [22].

Concluding, it is natural to put forward a conjecture that the spectrum of a spin system associated with the Hecke *R*-matrix with minimal rank of two projectors greater than 1 is also defined by the corresponding fusion procedure and the functional relations.

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References

[1] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251
[2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[3] Kauffman L 1991 *Knots and Physics* (Singapore: World Scientific)
[4] Faddeev L D, Reshetikhin N Y and Takhtajan L A 1989 *Algebra Anal.* **1** 178–206
Faddeev L D, Reshetikhin N Y and Takhtajan L A 1990 *Leningrad Math. J.* **1** 193 (Engl. Transl.)
[5] Faddeev L D 1998 How algebraic Bethe ansatz works for integrable model *Quantum Symmetries* (*Les Houches Session* vol 64) (Amsterdam: Elsevier) pp 149–219 (*Preprint* hep-th/9605187)
[6] Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* vol 151 (Berlin: Springer) p 69
[7] Barber M N and Batchelor M T 1989 *Phys. Rev.* B **40** 4621
[8] Batchelor M T and Hammer C J 1990 *J. Phys. A: Math. Gen.* **23** 761
[9] Klumper A 1990 *J. Phys. A: Math. Gen.* **23** 809
[10] Martins M J and Ramos P B 1994 *J. Phys. A: Math. Gen.* **27** L703
[11] Batchelor M T, de Gier J, Links J and Maslen M 2000 *J. Phys. A: Math. Gen.* **33** L97
[12] Koberle R and Lima-Santos A 1994 *J. Phys. A: Math. Gen.* **27** 5409
[13] Lima-Santos A 1998 *Nucl. Phys.* B **522** 503
[14] Kulish P P, Reshetikhin N Y and Sklyanin E K 1981 *Lett. Math. Phys.* **5** 393
[15] Kirillov A N and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1587
[16] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* A **9** 5215
[17] Reshetikhin N Y 1983 *Lett. Math. Phys.* **7** 205
[18] Ivanov B A and Kolezhuk A K 2002 *Phys. Rev.* B **51** 1121 (*Preprint* cond-mat/0205207)
[19] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
[20] Kulish P P 1993 *Theor. Math. Phys.* **94** 193
[21] Levi D and Martin P 1994 *J. Phys. A: Math. Gen.* **27** L521
[22] Kulish P P and Sklyanin E K 1991 *J. Phys. A: Math. Gen.* **24** L435